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Some Remarks on a Generalization of the Superintegrable Chiral Potts Model

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Abstract The spontaneous magnetization of a two-dimensional lattice model can be expressed in terms of the partition function W of a system with fixed boundary spins and an extra weight dependent on the value of a particular central spin. For the superintegrable case of the chiral Potts model with cylindrical boundary conditions, W can be expressed in terms of reduced Hamiltonians H and a central spin operator S. We conjectured in a previous paper that W can be written as a determinant, similar to that of the Ising model. Here we generalize this conjecture to any Hamiltonians that satisfy a more general Onsager algebra, and give a conjecture for the elements of S.

Keywords Statistical mechanics · Lattice models · Transfer matrices

1 Introduction

Onsager calculated the partition function of the two-dimensional Ising model by noting that the two Hamiltonians associated with the transfer matrices generated a finite-dimensional algebra, now known as the Onsager algebra [1, (60), (61)]. Later, Kaufman showed the problem could be solved by using free-fermion (i.e. Clifford algebra) operators [2]. This method leads naturally to determinantal expressions, and indeed Kac and Ward showed that the partition function could be expressed combinatorially as a determinant [3], while Hurst and Green [4] wrote it as a Pfaffian (the square root of an anti-symmetric determinant). Later it was realized that the Ising model could be expressed as a dimer problem, giving a direct combinatorial solution in terms of Pfaffians [5–8].

The calculation of the spontaneous magnetization $\mathcal{M}_0$ is a more difficult problem. Onsager announced his and Kaufman's result for the $\mathcal{M}_0$ in 1949 [9]. The first published proof was by Yang in 1952 [10]. Then in 1963, Montroll, Potts and Ward [11] showed that this problem could also be solved combinatorially in terms of determinants. To do this, one begins

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by writing $\mathcal{M}_0$ as

$$\mathcal{M}_0 = W/Z, \tag{1.1}$$

where W, Z are two partition functions (with open, fixed spin boundary conditions). Z is the usual partition function, while W is the partition function with an extra weight σ_0. Here σ_0 is the spin on a site 0 deep inside the lattice. In [11] Z, W are evaluated as determinants.

Like the Ising model, the general solvable N-state chiral Potts model is a solvable model. It has $N-1$ single-site order parameters (spontaneous magnetizations) $\mathcal{M}_r$, where $r = 1, \ldots, N-1$. Its transfer matrices satisfy the star-triangle relation [12]. It is, however, much more difficult mathematically. Its free energy (the logarithm of the partition function) was calculated in 1988 [13], but it was not until 2005 that $\mathcal{M}_r$ was calculated by solving certain functional relations derived from the star-triangle relation [14]. The calculation verified a long-standing conjecture of Albertini et al. [15].

The superintegrable chiral Potts model is a special case of the general solvable chiral Potts model. It has the same order parameters, so to obtain $\mathcal{M}_r$ for the general model it would be sufficient to obtain it for the superintegrable case.

Further, the superintegrable case has mathematical properties quite similar to those of the Ising model. The Hamiltonians $\mathcal{H}_0, \mathcal{H}_1$ associated with the transfer matrices also satisfy the Onsager algebra. If one imposes cylindrical boundary conditions, with fixed-spin open boundary conditions on the top and bottom of the lattice, then we show in Sect. 2 that $Z = u^\dagger DUu$, where the vectors $u^\dagger, u$ are determined by the bottom and top boundary conditions, and D, U can be taken to be exponentials of the Hamiltonian $\mathcal{H} = \mathcal{H}_0 + k'\mathcal{H}_1$ that commutes with the transfer matrix. Also, $W = W(r) = u^\dagger D\mathcal{S}_r Uu$, where the matrix $\mathcal{S}_r$ arises from the extra weight factor $\omega^{r\zeta}$ in (2.2). There is a reduced representation in which D, U are direct products of two-by-two matrices, as in the Ising model, and one can define a reduced form S_{PQ} of the matrix $\mathcal{S}_r$ by (3.31).

We recently conjectured [16] that $W(r)$ can be written as a determinant. As yet we have neither proved this conjecture, nor used it to obtain $\mathcal{M}_r$, but numerical studies strongly suggest that both the conjecture, and the resulting formula for $\mathcal{M}_r$, are correct.

Here we obtain commutation relations for S_{PQ} in terms of the reduced Hamiltonians H_0, H_1. We generalize the problem to one in which H_0, H_1 satisfy a quite general Onsager algebra, not just that of the superintegrable chiral Potts model.

The commutation relations appear to determine S_{PQ}. We conjecture their solution and the resulting determinantal form of $W(r)$. Our expectation is that these generalized conjectures will be easier to establish than the previous particular one.

2 Partition Function

Definition

We use the notation of [16] and define the N-state chiral Potts on the square lattice $\mathcal{L}$, rotated through 45°, with $M+1$ horizontal rows, each containing L spins, as in Fig. 1.

We impose cylindrical boundary conditions, so that the last column L is followed by the first column 1. At each site i there is a spin σ_i, taking the values $0, 1, \ldots, N-1$. The spins in the bottom row are fixed to have value a, those in the top row to have value 0. Adjacent spins σ_i, σ_j on southwest to northeast edges (with i below j) interact with Boltzmann weight $\overline{\mathcal{W}}(\sigma_i - \sigma_j)$; those on southeast to northwest edges with weight $\overline{\mathcal{W}}(\sigma_i - \sigma_j)$.

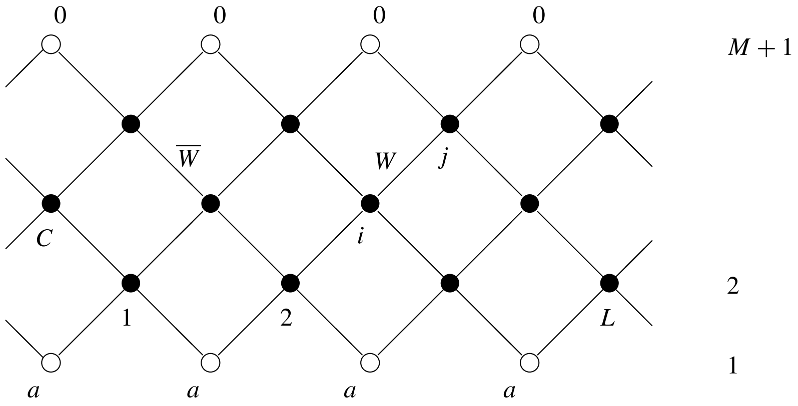

Fig. 1 The square lattice $\mathcal{L}$ turned through 45°

The partition function, which depends on a, is

$$Z_a = \sum_\sigma \prod_{\langle i,j\rangle} \mathcal{W}(\sigma_i - \sigma_j) \prod_{\langle i,j\rangle} \overline{\mathcal{W}}(\sigma_i - \sigma_j), \tag{2.1}$$

the products being over all edges of the two types. The sum is over all values of all the free spins.

To define the order parameter, we select some inner site C of $\mathcal{L}$, say the first site of row $j+1$. Then there are j rows of edges below C and $M-j$ above. Let ζ be the spin on site C and define

$$W_a(r) = \sum_\sigma \omega^{r\zeta} \prod_{\langle i,j\rangle} \mathcal{W}(\sigma_i - \sigma_j) \prod_{\langle i,j\rangle} \overline{\mathcal{W}}(\sigma_i - \sigma_j), \tag{2.2}$$

where

$$\omega = \mathrm{e}^{2\pi \mathrm{i}/N}, \quad 0 \le r \le N. \tag{2.3}$$

Then the order parameter is

$$\mathcal{M}_r = W_0(r)/Z_0, \tag{2.4}$$

evaluated in the limit when $L, j, M-j \to \infty$.

Transfer Matrices and Hamiltonians

As in [16], we define a vector u_a, of dimension N^L, with entries

$$\begin{aligned} (u_a)_\sigma &= 1 \quad \text{if } \sigma_1 = \cdots = \sigma_L = a, \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{2.5}$$

We also define a diagonal N^L by N^L matrix $\mathcal{S}_r$ with elements

$$(\mathcal{S}_r)_{\sigma,\sigma'} = \omega^{r\sigma_1} \prod_{j=1}^{L} \delta(\sigma_j, \sigma'_j). \tag{2.6}$$

We take $0 \le r \le N$.

Let T be the N^L by N^L transfer matrix, defined as in [16], let j be the number of rows below C, $M-j$ the number above, and set

$$D = T^j, \qquad U = T^{M-j}. \tag{2.7}$$

Then in the usual way, it follows that

$$Z_a = u_a^\dagger D U u_0, \qquad W_a(r) = u_a^\dagger D \mathcal{S}_r U u_0. \tag{2.8}$$

The transfer matrix T commutes with a Hamiltonian $\mathcal{H}$. For simplicity, we replace the definitions (2.7) by

$$D = \mathrm{e}^{-\alpha\mathcal{H}}, \qquad U = \mathrm{e}^{-\beta\mathcal{H}}. \tag{2.9}$$

For the ferromagnetic model, we expect $\mathcal{M}_r$ to be unchanged if we now define it by (2.9), (2.8), (2.4) and take the limit $\alpha, \beta, L \to +\infty$.

Superintegrable Case

Let

$$\omega = \exp^{2\pi \mathrm{i}/N} \tag{2.10}$$

and, as in [15], define N^L by N^L matrices Z_j, X_j by

$$\begin{aligned} (Z_j)_{\sigma,\sigma'} &= \omega^{\sigma_j} \prod_{m=1}^{L} \delta(\sigma_m, \sigma'_m), \\ (X_j)_{\sigma,\sigma'} &= \delta(\sigma_j, \sigma'_j + 1) \prod_{n=1}^{L}{}^{*} \delta(\sigma_n, \sigma'_n), \end{aligned} \tag{2.11}$$

the $*$ on the last product indicating that it excludes the case $n = j$. Then from (2.6)

$$\mathcal{S}_r = Z_1^r. \tag{2.12}$$

For the general solvable chiral Potts model, the Hamiltonian $\mathcal{H}$ is given by Albertini et al. [15] as a linear combination of the matrices $Z_j^n Z_{j+1}^{-n}$ and of X_j^N. For the superintegrable case (in their notation $\phi = \overline{\phi} = \pi/2$) this becomes (writing their λ as k')

$$\mathcal{H} = \mathcal{H}_0 + k'\mathcal{H}_1, \tag{2.13}$$

where

$$\begin{aligned} \mathcal{H}_0 &= -2 \sum_{j=1}^{L} \sum_{n=1}^{N-1} \frac{Z_j^n Z_{j+1}^{-n}}{1-\omega^{-n}}, \\ \mathcal{H}_1 &= -2 \sum_{j=1}^{L} \sum_{n=1}^{N-1} \frac{X_j^n}{1-\omega^{-n}}. \end{aligned} \tag{2.14}$$

The k' in (2.13) is a "temperature-like" parameter, satisfying

$$0 < k' < 1 \tag{2.15}$$

in the ferromagnetic regime, being small at low temperatures, and tending towards one as the system becomes critical.

Onsager Algebra

These Hamiltonians generate the "Onsager algebra" [1, (60), (61)] and [17–19]. Define

$$A_0 = -2\mathcal{H}_1/N, \qquad A_1 = 2\mathcal{H}_0/N. \tag{2.16}$$

Then there are two sets of matrices A_m, G_n such that

$$\begin{aligned} &[A_m, A_n] = 4G_{m-n}, \\ &[G_m, A_n] = 2A_{m+n} - 2A_{n-m}, \qquad [G_m, G_n] = 0, \end{aligned} \tag{2.17}$$

for all integers m, n.

The matrices $\mathcal{H}_0$, $\mathcal{H}_1$ have a highly degenerate eigenvalue structure. Note that

$$-\sum_{n=1}^{N-1} \frac{2\omega^{kn}}{1-\omega^{-n}} = 2k + 1 - N, \quad 0 \le k < N \tag{2.18}$$

so the LHS is a "sawtooth" function, periodic of period N, linear from $k = 0$ to $k = N - 1$.

The matrices Z_j are diagonal, and $Z_j^n Z_{j+1}^{-n}$ has entries $\omega^{n(\sigma_j - \sigma_{j+1})}$. It follows that the diagonal elements of $\mathcal{H}_0$ are of the form

$$L(1-N) + 2mN, \tag{2.19}$$

where m is an integer and

$$0 \le m \le L(N-1)/N.$$

There is a similarity transformation that takes X_j to Z_j. It follows that the eigenvalues of $\mathcal{H}_1$ are also integers of the form (2.19), though with different degeneracies from those of $\mathcal{H}_0$.

Commutators with $\mathcal{S}_r$

To evaluate the matrix elements (2.8), we look at the matrices formed by setting $C_1 = \mathcal{S}_r$ (for a given value of r) and then looking at the sequence of C_m generated by successively forming the commutators $[\mathcal{H}_0, C_m]$ and $[\mathcal{H}_1, C_m]$.

It is convenient to define linear operators f_0, f_1 by

$$f_0(C) = \frac{[\mathcal{H}_0, C]}{2N}, \qquad f_1(C) = \frac{[\mathcal{H}_1, C] + 2rC}{2N} \tag{2.20}$$

for any N^L-dimensional matrix C.

We first note from (2.12), (2.13) that S_r, $\mathcal{H}_0$ are diagonal matrices, so S_r commutes with $\mathcal{H}_0$, so

$$f_0(C_1) = 0. \tag{2.21}$$

We can therefore start by forming all the linearly independent commutators with $\mathcal{H}_1$. If we define

$$C_2 = f_1(C_1),$$

then we prove in Appendix A that

$$f_1(C_2) = C_2 \tag{2.22}$$

so we now have two matrices C_1, C_2. They are in general linearly independent.

We have proceeded by performing numerical experiments for small N, L and now report our observations.

The next step is to form all possible commutators with $\mathcal{H}_0$. This leads us to define two more matrices:

$$C_3 = f_0(C_2), \qquad C_4 = f_0(C_3),$$

and we find that

$$f_0(C_4) = C_3. \tag{2.23}$$

So at this stage we have four matrices, satisfying the three relations (2.21), (2.22), (2.23).

Now we commute with $\mathcal{H}_1$, defining four new matrices:

$$C_5 = f_1(C_3), \qquad C_6 = f_1(C_4),$$
$$C_7 = f_1(C_6), \qquad C_8 = f_1(C_7) - C_6,$$

and find two relations:

$$f_1(C_5) = C_5, \qquad f_1(C_8) = 2C_8, \tag{2.24}$$

giving eight matrices and five relations in all.

If we now form all commutators with $\mathcal{H}_0$, we find eight new matrices:

$$C_9 = f_0(C_5), \qquad C_{10} = f_0(C_6), \qquad C_{11} = f_0(C_7),$$
$$C_{12} = f_0(C_{11}), \qquad C_{13} = f_0(C_8), \qquad C_{14} = f_0(C_{13}),$$
$$C_{15} = f_0(C_{14}) - C_{13}, \qquad C_{16} = f_0(C_{15}),$$

with four relations:

$$f_0(C_{10}) = C_9, \qquad f_0(C_9) = C_{10}, \qquad f_0(C_{12}) = C_{11}, \qquad f_0(C_{16}) = 4C_{15}, \tag{2.25}$$

a total of 16 matrices and 9 relations.

At each stage we have a total of 2^m matrices (linearly independent provided L is sufficiently large), satisfying a total of $1 + 2^{m-1}$ relations, for $m = 1, 2, 3, 4$. Our numerical studies support the conjecture that this pattern continues for all integers m and all N, L, r such that $0 < r < N$.

3 Reduced Representation

Both $\mathcal{H}_0$ and $\mathcal{H}_1$ commute with the matrix

$$R = X_1 X_2 \cdots X_L \tag{3.1}$$

which satisfies $R^N = 1$ and has eigenvalues $1, \omega, \ldots, \omega^{N-1}$. If

$$v_P = N^{-1/2} \sum_{a=0}^{N-1} \omega^{-Pa} u_a \tag{3.2}$$

for $P=0,1,\dots,N-1$, then

$$R v_P = \omega^P v_P. \tag{3.3}$$

The full N^L-dimensional space is the union of N sub-spaces $\mathcal{V}_0, \mathcal{V}_1, \dots, \mathcal{V}_{N-1}$ such that

$$R v = \omega^P v \quad \text{if } v \in \mathcal{V}_P \tag{3.4}$$

and if two vectors v, w belong to different sub-spaces, then

$$v^\dagger w = 0. \tag{3.5}$$

Clearly $v_P \in \mathcal{V}_P$, and, because $\mathcal{H}$ commutes with R,

$$DU v_P \in \mathcal{V}_P, \qquad D\mathcal{S}_r U v_Q \in \mathcal{V}_P, \tag{3.6}$$

where

$$Q = P + r \pmod N. \tag{3.7}$$

From (2.8) and (2.9), Z_a is a function of $\alpha+\beta$, and $W_a(r)$ of α,β separately. We define

$$\tilde{Z}_P(\alpha+\beta) = \sum_{a=0}^{N-1} \omega^{Pa} Z_a, \qquad \tilde{W}_{PQ}(\alpha,\beta) = \sum_{a=0}^{N-1} \omega^{Pa} W_a(r) \tag{3.8}$$

and it then follows that

$$\tilde{Z}_P(\alpha+\beta) = v_P^\dagger DU v_P, \qquad \tilde{W}_{PQ}(\alpha,\beta) = v_P^\dagger D\mathcal{S}_r U v_Q, \tag{3.9}$$

where P, Q are again related by (3.7).

The author observed [20] that if one pre-multiplies the vector v_P by various transfer matrices T (in general with different values of the horizontal rapidity), then one does not generate the full vector space $\mathcal{V}_P$, but a smaller space V_P in which T has 2^m distinct eigenvalues, where

$$m = m(P) = \left[\frac{(N-1)L - P}{N} \right] \tag{3.10}$$

and $[x]$ means the integer part of x. Each eigenvalue occurs only once.

Label the basis vectors of V_P by

$$s = \{s_1, s_2, \dots, s_m\}, \tag{3.11}$$

where each s_i takes the values 0 or 1. (We can think of each $1-2s_i$ as an "Ising spin", with value ± 1.) Thus there are 2^m vectors $\tilde{v}_s = \tilde{v}(s_1, s_2, \dots, s_m)$, each of dimension N^L. We define

$$\kappa_s = s_1 + s_2 + \cdots + s_m \tag{3.12}$$

so κ_s is an integer, and

$$0 \le \kappa_s \le m.$$

In [21] we showed that we could choose the vectors $\tilde{v}_s$ so that

$$v_P = \tilde{v}(0,0,\ldots,0), \tag{3.13}$$

$$\mathcal{H}\tilde{v}_s = \mu_P \tilde{v}_s - N\sum_{j=1}^{m}(1-k'\cos\theta_j)s_j\tilde{v}_s$$
$$+ Nk'\sum_{j=1}^{m}\sin\theta_j\tilde{v}(s_1,\ldots,-s_j,\ldots,s_m), \tag{3.14}$$

where

$$\mu_P = 2k'P + (1+k')(mN - NL + L). \tag{3.15}$$

What we did not show, but believe to be true, is that the vectors $\tilde{v}_s$ can all be chosen to be independent of k'. For small N, L we can generate these vectors algebraically on the computer, and find this to be so. This is consistent with the fact that $\mathcal{H}$ is linear in k' [22].

Define 2^m by 2^m matrices $\hat{S}_j, \hat{C}_j$ by

$$(\hat{S}_j)_{s,s'} = s_j\prod_{n=1}^{m}\delta(s_n, s'_n), \tag{3.16}$$

$$(\hat{C}_j)_{s,s'} = \delta(s_j, 1-s'_j)\prod_{n=1}^{m}{}^{*}\delta(s_n, s'_n), \tag{3.17}$$

where again the $*$ means that the term $n=j$ is excluded from the product. Then from (3.14), with respect to the basis vectors $\tilde{v}_s$, the Hamiltonian $\mathcal{H}$ is now

$$H = H_0 + k'H_1, \tag{3.18}$$

where

$$H_0 = L - NL + 2NJ_0, \tag{3.19}$$

$$H_1 = 2P + L - NL + 2NJ_1, \tag{3.20}$$

and

$$2J_0 = mI - \sum_{j=1}^{m}\hat{S}_j, \tag{3.21}$$

$$2J_1 = mI + \sum_{j=1}^{m}(\cos\theta_j\hat{S}_j + \sin\theta_j\hat{C}_j), \tag{3.22}$$

I being the identity matrix of dimension 2^m. The reduced Hamiltonians H, H_0, H_1, J_0, J_1 are also of dimension 2^m. If we replace $\mathcal{H}_0, \mathcal{H}_1$ in (2.16) by H_0, H_1, then again we obtain the Onsager algebra (2.17).

In this basis we see from (3.13) that v_P is replaced by the 2^m-dimensional vector v_P with entries

$$(\mathrm{v}_P)_s = 1 \quad \text{if } s = \{0,0,\ldots,0\},$$
$$= 0 \quad \text{else}, \tag{3.23}$$

i.e.

$$\mathrm{v}_P = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.24}$$

The vectors $\tilde{v}_s$ depend on P, so where necessary we write them as $\tilde{v}_s^P$. Similarly we may write m, θ_j, H, H_0, H_1 as $m(P), \theta_j^P, H_P, H_0^P, H_1^P$. In particular, we consider two particular values P, Q of the index P, related by (3.7), and set

$$m = m(P), \qquad \theta_i = \theta_i^P; \qquad n = m(Q), \qquad \theta_j' = \theta_j^Q, \tag{3.25}$$

where $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

We have not yet defined the $\theta_1, \ldots, \theta_m$ (and $\theta_1', \ldots, \theta_n'$). This is because we believe the equations of this paper to apply for *arbitrary* $\theta_1, \ldots, \theta_m$ and $\theta_1', \ldots, \theta_n'$. We do not use the definitions here, but for completeness they are given in Appendix B.

Calculation of $\tilde{Z}_P$, $\tilde{W}_{PQ}$

The function $\tilde{Z}_P$ is unchanged if we replace $\mathcal{H}$, v_P in (3.9), (2.9) by the reduced matrices and vectors H, v_P. The exponential $\mathrm{e}^{-\alpha H}$ is a direct product of two-by-two matrices, so is easily calculated. As in (3.16) of [16], define functions $\lambda(\theta), u(\alpha, \theta), v(\alpha, \theta), w(\alpha, \theta)$ by

$$\lambda(\theta) = \lambda = (1 - 2k' \cos\theta + k'^2)^{1/2}, \tag{3.26}$$

$$\begin{aligned} u(\alpha, \theta) &= \cosh(N\alpha\lambda) + \frac{1 - k'\cos\theta}{\lambda} \sinh(N\alpha\lambda), \\ v(\alpha, \theta) &= -\frac{k'\sin\theta}{\lambda} \sinh(N\alpha\lambda), \\ w(\alpha, \theta) &= \cosh(N\alpha\lambda) - \frac{1 - k'\cos\theta}{\lambda} \sinh(N\alpha\lambda), \end{aligned} \tag{3.27}$$

and let U_j be the two-by-two matrix

$$U_j = \begin{pmatrix} u_p(\alpha, \theta_j) & v_p(\alpha, \theta_j) \\ v_p(\alpha, \theta_j) & w_p(\alpha, \theta_j) \end{pmatrix}, \tag{3.28}$$

then

$$\mathrm{e}^{-\alpha H} = e^{-\mu_P \alpha} U_1 \otimes U_2 \otimes \cdots \otimes U_m. \tag{3.29}$$

From (3.9), it follows that

$$\tilde{Z}_P(\alpha) = \mathrm{e}^{-\mu_P \alpha} u_P(\alpha, \theta_1) \cdots u_P(\alpha, \theta_m). \tag{3.30}$$

We can similarly write down an expression for of $\tilde{W}_{PQ}$, provided we replace v_P by v_P, v_Q by v_Q, the $\mathcal{H}$ in D by H_P, the $\mathcal{H}$ in U by H_Q, and $\mathcal{S}_r$ by a reduced matrix S_{PQ} with elements

$$(S_{PQ})_{s,s'} = (\tilde{v}_s^P)^\dagger \mathcal{S}_r \tilde{v}_{s'}^Q. \tag{3.31}$$

Note that the set s has m entries, while s' has n. Hence the reduced matrix S_{PQ} is of dimension 2^m by 2^n. It is not necessarily square.

Define

$$x_i = \frac{v(\alpha, \theta_i)}{u(\alpha, \theta_i)}, \qquad x'_i = \frac{v(\beta, \theta'_i)}{u(\beta, \theta'_i)}. \tag{3.32}$$

Then we obtain

$$\tilde{W}_{PQ}(\alpha, \beta) = Z_P(\alpha) Z_Q(\beta) \mathcal{D}_{PQ}, \tag{3.33}$$

where

$$\mathcal{D}_{PQ} = \sum_s \sum_{s'} x_1^{s_1} x_2^{s_2} \cdots x_m^{s_m} \left(S_{PQ}\right)_{s,s'} x_1'^{s'_1} x_2'^{s'_2} \cdots x_n'^{s'_n}. \tag{3.34}$$

However, this is still a 2^{m+n}-dimensional summation. In the following sections we firstly give an explicit conjecture for $(S_{PQ})_{s,s'}$, and secondly a conjectured expression for $\mathcal{D}_{PQ}$ as an m by m (or n by n) determinant. The formula (3.34) is the same as (5.37) of [16], but now the θ_j, θ'_j are arbitrary.

The Commutators

Multiply any of (2.20)–(2.25) on the left by the hermitian conjugate of an arbitrary vector $\tilde{v}_s^P$ of the P-set, and on the right by a vector $\tilde{v}_{s'}^Q$ of the Q-set. If we define reduced matrices $C_1, \ldots, C_{16}$ analogously to (3.31), then we see that (2.20)–(2.25) remain valid if we replace each C_j by its reduced form, and any $\mathcal{H}_0$ to the left (right) of the C matrix by H_0^P (H_0^Q) and $\mathcal{H}_1$ by H_1^P (H_1^Q).

We can use (3.21), (3.22) to replace $H_0^P, \ldots, H_1^Q$ in these commutation relations by $J_0^P, \ldots, J_1^Q$. We have to take care to note that $0 \le P, Q < N$ and $r = Q - P, \bmod N$, so $0 < r < N$. The general commutators (2.20) become

$$f_0(C) = J_0^P C - C J_0^Q, \qquad f_1(C) = J_1^P C - C J_1^Q + \frac{1-\gamma}{2} C, \tag{3.35}$$

where

$$\gamma = 1 \quad \text{if } P < Q, \qquad \gamma = -1 \quad \text{if } P > Q. \tag{3.36}$$

From (3.10), there are four possible cases to consider. We define a function $e(P, Q, i)$ in each case as follows.

$$\begin{aligned}
&1) \ e(P,Q,i) = \sin\theta_i && \text{if } P < Q,\ n = m-1,\ \gamma = 1,\\
&2) \qquad\qquad\ \ = \tan(\theta_i/2) && \text{if } P < Q,\ n = m,\quad\ \ \gamma = 1,\\
&3) \qquad\qquad\ \ = 1/\sin\theta_i && \text{if } P > Q,\ n = m+1,\ \gamma = -1,\\
&4) \qquad\qquad\ \ = \cot(\theta_i/2) && \text{if } P > Q,\ n = m,\quad\ \ \gamma = -1.
\end{aligned} \tag{3.37}$$

Similarly,

$$e(Q, P, i) = 1/\sin\theta'_i, \cot(\theta'_i/2), \sin\theta'_i, \tan(\theta'_i/2) \tag{3.38}$$

for cases $1, \ldots, 4$, respectively.

In the rest of this paper we take the $\theta_i, \theta'_i, x_i, y_i$ to be arbitrary and will no longer use the relation (3.10) between N, L, P, m, or between N, L, Q, n. However, we stress that the restrictions (3.37) appear to be necessary: in particular, we have not found any generalizations to $n > m+1$ or $n < m-1$.

The Reduced Matrix S_{PQ}

Using (3.35), (3.36), we obtain two equations for S_{PQ}, namely

$$J_0^P S_{PQ} = S_{PQ} J_0^Q, \tag{3.39}$$

and

$$J_1^P J_1^P S_{PQ} - 2 J_1^P S_{PQ} J_1^Q + S_{PQ} J_1^Q J_1^Q = \gamma (J_1^P S_{PQ} - S_{PQ} J_1^Q). \tag{3.40}$$

These equations do not determine the normalization of S_{PQ}. To do this we note from (2.6), (3.2), (3.13) that

$$(S_{PQ})_{s,s'} = 1 \quad \text{if } s = \mathbf{0} \quad \text{and} \quad s' = \mathbf{0}'. \tag{3.41}$$

Here $\mathbf{0} = \{0, 0, \ldots, 0\}$ has m entries and $\mathbf{0}' = \{0, 0, \ldots, 0\}$ has n entries.

These give two commutation relations for S_{PQ}. The first is simple. From (3.21), J_0^P is a diagonal matrix with entries

$$0, 1, 2, \ldots, m$$

and degeneracies $1, m, m(m-1)/2, \ldots$. If we order the rows and columns of J_0^P and J_0^Q so that the diagonal entries are in increasing order, then (3.39) implies that S_{PQ} is block-diagonal. More generally, (3.39) implies that

$$(S_{PQ})_{s,s'} = 0 \quad \text{unless } \kappa_s = \kappa_{s'}. \tag{3.42}$$

The second (double) commutation relation is more complicated, but algebraic computer calculations for small m, n satisfying (3.37) strongly suggest that

a) the relations (3.39)–(3.41) uniquely determine S_{PQ}.
b) the non-zero elements of S_{PQ} are simple products.

To formulate our observations more specifically, we first need some further definitions. For a given set s, let V be the set of integers i such that $s_i = 0$ and W the set such that $s_i = 1$. Hence, from (3.12), V has $m - \kappa_s$ elements, while W has κ_s. Define V', W' similarly for the set s'. Set

$$c_i = \cos\theta_i, \qquad c'_j = \cos\theta'_j, \tag{3.43}$$

for $1 \le i \le m$ and $1 \le j \le n$. Let

$$\begin{aligned}
A_{s,s'} &= \prod_{i\in W} \prod_{j\in V'} (c_i - c'_j), \qquad & B_{s,s'} &= \prod_{i\in V} \prod_{j\in W'} (c_i - c'_j), \\
C_s &= \prod_{i\in W} \prod_{j\in V} (c_j - c_i), \qquad & D_{s'} &= \prod_{i\in V'} \prod_{j\in W'} (c'_j - c'_i), \\
\mathcal{T}_s &= \prod_{i\in W} e(P, Q, i), \qquad & \mathcal{T}'_{s'} &= \prod_{i\in W'} e(Q, P, i).
\end{aligned} \tag{3.44}$$

Then our calculations are consistent with the conjecture

$$(S_{PQ})_{s,s'} = \frac{\mathcal{T}_s \mathcal{T}'_{s'} A_{s,s'} B_{s,s'}}{C_s D_{s'}}, \tag{3.45}$$

when $\kappa_s = \kappa_{s'}$, for all four cases (3.37). This agrees with the symmetry

$$S_{PQ} = \left(S_{QP}\right)^{\dagger}, \tag{3.46}$$

which follows from (2.6) and (3.31). If we define

$$y_i = e(P, Q, i)x_i, \qquad y'_i = e(Q, P, i)x'_i, \tag{3.47}$$

it implies that (3.34) can be written

$$\mathcal{D}_{PQ} = \sum_s \sum_{s'} y_1^{s_1} y_2^{s_2} \cdots y_m^{s_m} \left(\frac{A_{s,s'} B_{s,s'}}{C_s D_{s'}}\right) y_1'^{\,s'_1} y_2'^{\,s'_2} \cdots y_n'^{\,s'_n}, \tag{3.48}$$

the sum being restricted to s, s' such that $\kappa_s = \kappa_{s'}$.

4 Determinantal Conjecture

We emphasize that (3.45) is independent of the definitions (B.2) of the θ_i and θ'_i, so should apply for arbitrary θ_i, θ'_i. In [16] we conjectured that $\tilde{W}_{PQ}$ could be written as a determinant, and this result also appears to be true for arbitrary θ_i, θ'_i. We repeat it here for this generalization.

Define two functions $\mathcal{P}_P(c), \mathcal{P}_Q(c)$ by

$$\mathcal{P}_P(c) = \prod_{i=1}^{m}(c - \cos\theta_i), \qquad \mathcal{P}_Q(c) = \prod_{i=1}^{n}(c - \cos\theta'_i). \tag{4.1}$$

They are polynomials in c, of degree m, n, respectively. Let

$$\Delta_P(c) = \frac{\mathrm{d}}{\mathrm{d}c}\mathcal{P}_P(c) \tag{4.2}$$

and similarly for $\Delta_Q(c)$. Let

$$\epsilon(P, Q) = 1 \quad \text{if } P < Q, \qquad \epsilon(P, Q) = -1 \quad \text{if } P > Q \tag{4.3}$$

and define functions

$$f(P, Q, c) = \left[\epsilon(P, Q)\mathcal{P}_Q(c)/\Delta_P(c)\right]^{1/2}, \tag{4.4}$$

$$\mathcal{B}(P, Q, c, c') = \frac{f(P, Q, c) f(Q, P, c')}{c - c'} \tag{4.5}$$

for $P \neq Q$. They are rational functions of c, c'.

Let B_{PQ} be the m by n matrix with elements

$$\left(B_{PQ}\right)_{ij} = \mathcal{B}(P, Q, \cos\theta_i, \cos\theta'_j). \tag{4.6}$$

By construction it is orthogonal, in the sense that

$$B_{PQ}^T B_{PQ} = I \quad \text{if } m \geq n, \qquad B_{PQ} B_{PQ}^T = I \quad \text{if } m \leq n. \tag{4.7}$$

Define the n by m matrix B_{QP} similarly, with P, m, θ_i interchanged with Q, n, θ'_i, respectively. Also define an m by m diagonal matrix Y_{PQ}, and an n by n diagonal matrix Y_{QP}, with elements

$$(Y_{PQ})_{i,j} = y_i \delta_{i,j}, \qquad (Y_{QP})_{i,j} = y'_i \delta_{i,j}. \tag{4.8}$$

We conjecture that

$$\mathcal{D}_{PQ} = \det[I_m - Y_{PQ} B_{PQ} Y_{QP} B_{QP}] \tag{4.9}$$

or equivalently

$$\mathcal{D}_{PQ} = \det[I_n - Y_{QP} B_{QP} Y_{PQ} B_{PQ}]. \tag{4.10}$$

Here I_m (I_n) is the identity matrix, of dimension m (n).

These equations (4.9), (4.10) are the same as (7.2), (7.3) of [16] when θ_i, θ'_i are given as in Appendix B.

5 Summary

If we consider the superintegrable chiral Potts model with cylindrical boundary conditions, and fixed equal spins in the top and bottom rows, we are led to the reduced Hamiltonians J_0^P, J_1^P given by (3.21), (3.22). The θ_i in (3.22) are given as in Appendix B, but for *all* θ_i it is true that if we take

$$A_0 = -4J_1^P, \qquad A_1 = 4J_0^P, \tag{5.1}$$

then we can define matrices A_m, G_m such that the Onsager algebra (2.17) is satisfied.

To calculate the spontaneous magnetization we must introduce the diagonal matrix $\mathcal{S}_r$ of (2.6). Its reduced form S_{PQ} of (3.31) satisfies (3.39), (3.40). Here we consider these equations for arbitrary θ_i, θ'_i and conjecture that, together with the normalization condition (3.41), they uniquely define (3.31), and that the solution is (3.45).

We show in [16] that the spontaneous magnetization is given by an expression of the general form (3.34). Here we take the x_i, x'_i therein to be arbitrary and define related quantities y_i, y'_i by (3.47). We then generalize our previous conjecture (7.2), (7.3) of [16] to (4.9), (4.10), still keeping the θ_i, θ'_i arbitrary (but note that m, n must satisfy the restrictions (3.37).

The factors $\mathcal{T}_s, \mathcal{T}'_{s'}$ can be removed from the equations (3.39), (3.40) by incorporating them into the J_0, J_1 expressions in (4.9), (4.10). We do this in Appendix C. Our conjectures then reduce to rational identities in the arbitrary variables c_i, c'_i. In this form they should be easier to establish.

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Appendix A

Here we prove the commutation relation (2.22). We take $0 < r < N$.

Since $\mathcal{S}_r = Z_1^r$ commutes with all the terms in the definition (2.14) of $\mathcal{H}_1$ except the $j = 1$ term, we can replace $\mathcal{H}_1$ in (2.14) by

$$\mathcal{H}_1 = -2\sum_{n=1}^{N-1} \frac{X_1^n}{1-\omega^{-n}}. \tag{A.1}$$

Also, the matrices Z_1, X_1 defined by (2.11) satisfy

$$Z_1 X_1 = \omega X_1 Z_1. \tag{A.2}$$

This relation is unchanged if we replace Z_1, X_1 by X_1^{-1}, Z_1, and indeed there is a similarity transformation that does this. Doing this and using the formula (2.18), it follows that for the purposes of this Appendix we can take $\mathcal{H}_1$ to be the N by N diagonal matrix

$$\mathcal{H}_1 = \begin{pmatrix} 1-N & 0 & \cdots & 0 \\ 0 & 3-N & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & N-1 \end{pmatrix} \tag{A.3}$$

and $\mathcal{S}_r$ to be the matrix whose elements (i, j) are zero unless $j = i + r \pmod N$ when they are one. We can therefore write $\mathcal{S}_r$ as

$$\mathcal{S}_r = A + B, \tag{A.4}$$

where A, B are the N by N matrices

$$A = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \qquad B = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \tag{A.5}$$

the $\mathbf{1}$ in the equation for A being the identity matrix of dimension $N - r$ and the $\mathbf{1}$ for B being of dimension r. All other elements of A and B are zero.

We readily see that

$$[\mathcal{H}_1, A] = -2rA, \qquad [\mathcal{H}_1, B] = 2(N-r)B. \tag{A.6}$$

Hence

$$[\mathcal{H}_1, \mathcal{S}_r] + 2r\mathcal{S}_r = 2NB, \qquad [\mathcal{H}_1, B] + 2rB = 2NB. \tag{A.7}$$

These relations are of course independent of similarity transformations. Setting $C_1 = \mathcal{S}_r$ and $C_2 = B$, we see that we have proved the relation (2.22).

Appendix B

For a given value of P with $0 \le P < N$, define a polynomial $\rho(w)$, of degree m, by

$$\rho(z^N) = z^{-P} \sum_{n=0}^{N-1} \omega^{(L+P)n} (z^N - 1)^L / (z - \omega^n)^L. \tag{B.1}$$

Let its zeros be $w_1, \ldots, w_m$ and define $\theta_1, \ldots, \theta_m$ by

$$\cos\theta_j = (1 + w_j)/(1 - w_j), \quad 0 < \theta_i < \pi, \tag{B.2}$$

for $j=1,\dots,m$. These are the θ's of the superintegrable chiral Potts model [16, 20]. They depend on L,N,P, so we may write θ_i as $\theta_{P,i}$. They are independent of k'. We do *not* use them in this paper. In particular our conjectures (3.45), (4.9), (4.10) are for arbitrary θ's.

Appendix C

Here we explicitly write the commutation relations (3.39), (3.40) in terms of matrices that are rational functions of $c_i=\cos\theta_i$, $c'_i=\cos\theta'_i$.

From (3.44), we are led to define a modified matrix $\tilde{S}_{PQ}$ by the equivalence transformation

$$S_{PQ}=E_{PQ}\tilde{S}_{PQ}E_{QP}, \tag{C.1}$$

where E_{PQ} is a direct product of m two-by two diagonal matrices:

$$E_{PQ}=\begin{pmatrix}1&0\\0&e_1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&e_2\end{pmatrix}\otimes\cdots\otimes\begin{pmatrix}1&0\\0&e_m\end{pmatrix}, \tag{C.2}$$

and $e_i=e(P,Q,i)$. The matrix E_{QP} is defined similarly, with m replaced by n and e_i replaced by $e'_i=e(Q,P,i)$. We also define $\tilde{J}_1^P$, $\tilde{J}_1^Q$ by[1]

$$J_1^P=E_{PQ}\tilde{J}_1^P E_{PQ}^{-1},\qquad J_1^Q=E_{QP}^{-1}\tilde{J}_1^Q E_{QP}. \tag{C.3}$$

For the four cases (3.37), let

$$\xi_i=1-c_i^2,\,1-c_i,\,1,\,1+c_i, \tag{C.4}$$

repectively, and set

$$\Gamma_i=\mathrm{e}\otimes\cdots\otimes\begin{pmatrix}0&\xi_i\\(1-c_i^2)/\xi_i&0\end{pmatrix}\otimes\cdots\otimes\mathrm{e}, \tag{C.5}$$

each e being the two-by-two identity matrix and the displayed matrix being in position i. Then

$$2\tilde{J}_1^P=mI+\sum_{j=1}^{m}(c_j\hat{S}_j+\Gamma_j). \tag{C.6}$$

It is a polynomial in $c_1,\dots,c_m$. The matrix $\tilde{J}_1^Q$ is also given by (C.4)–(C.6), but with m, c_i replaced by n, c'_i.

With these equivalence and similarity transformations, the commutation relations (3.39), (3.40) become

$$J_0^P\tilde{S}_{PQ}=\tilde{S}_{PQ}J_0^Q, \tag{C.7}$$

and

$$\tilde{J}_1^P\tilde{J}_1^P\tilde{S}_{PQ}-2\tilde{J}_1^P\tilde{S}_{PQ}\tilde{J}_1^Q+\tilde{S}_{PQ}\tilde{J}_1^Q\tilde{J}_1^Q=\gamma(\tilde{J}_1^P\tilde{S}_{PQ}-\tilde{S}_{PQ}\tilde{J}_1^Q)$$

[1] Each actually depends on both P and Q because of the restrictions (3.37).

and, from (3.45) and (C.1), our conjectured solution is

$$\left(\tilde{S}_{PQ}\right)_{s,s'} = \frac{A_{s,s'} B_{s,s'}}{C_s D_{s'}} \delta(\kappa_s, \kappa_{s'}). \tag{C.8}$$

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